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1995 J. Phys. A: Math. Gen. 28 5633

(http://iopscience.iop.org/0305-4470/28/19/017)

J. Phys. A: Math. Gen. 28 (1995) 5633–5642. Printed in the UK

A note on integrable two-degrees-of-freedom Hamiltonian systems with a second integral quartic in the momenta

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Received 3 January 1995, in final form 3 July 1995

Abstract. Two of the simplest integrable Hamiltonians $H(x, y, p_x, p_y) = (p_x^2 + p_y^2)/2 + V(x, y)$ with a second integral quartic in the momenta are those with potentials $V_3(x, y) = by(3x^2 + 16y^2) + d(x^2 + 16y^2) + \eta y$ and $V_4(x, y) = a(x^4 + 6x^2y^2 + 8y^4) + c(x^2 + 4y^2) + \nu y^{-2}$. We show how V_3 can be obtained from V_4. In the process we obtain a new potential of the class, V_N, that includes both V_3 and V_4 as particular cases. For this potential we give the second integral of motion, separating variables, a Lax representation and a bi-Hamiltonian structure, thus synthesizing the corresponding results for potentials V_3 and V_4. The integrable extension $V_N + \mu x^{-2}$ is also discussed.

1. Introduction

Two-degrees-of-freedom Liouville integrable Hamiltonian systems

$$\dot{x} = \frac{\partial H}{\partial p_x} \qquad \dot{y} = \frac{\partial H}{\partial p_y} \qquad \dot{p}_x = -\frac{\partial H}{\partial x} \qquad \dot{p}_y = -\frac{\partial H}{\partial y} \tag{1}$$

with a Hamiltonian function in 'natural' form

$$H(x, y, p_x, p_y) = T(p_x, p_y) + V(x, y) = \tfrac{1}{2}\left(p_x^2 + p_y^2\right) + V(x, y) \tag{2}$$

and a second integral of motion quartic in the momenta

$$I(x, y, p_x, p_y) = \sum_{\substack{m,n=0 \\ m+n\leqslant 4}}^{4} f_{mn}(x, y) p_x^m p_y^n \tag{3}$$

have been the object of several studies in the past few years (see the review by Hietarinta [1] and references therein; see also [2–9] for more recent work).

Probably the simplest systems of the above class are those with potentials [1]

$$\hat{V}_3(x, y) = by\left(3x^2 + 16y^2\right) + d\left(x^2 + 16y^2\right)$$
$$\hat{V}_4(x, y) = a\left(x^4 + 6x^2y^2 + 8y^4\right) + c\left(x^2 + 4y^2\right)$$

where a, b, c and d are constant parameters. $\hat{V}_3$ is one of the three integrable cases of the Hénon–Heiles system while $\hat{V}_4$ is one the four integrable cases of the two-degrees-of-freedom system with a quartic potential [1]. The corresponding second integrals of motion are

$$\hat{I}_3(x, y, p_x, p_y) = \left\{p_x^2 + 2x^2(by + d)\right\}^2 - 4bx^2 p_x(xp_y - 2yp_x) - 16bx^4 y(by + d) - 2b^2x^6$$

$$\hat{I}_4(x, y, p_x, p_y) = \left\{p_x^2 + 2x^2\left[a(x^2 + 2y^2) + c\right]\right\}^2 + 4ax^2(xp_y - 2yp_x)^2 .$$

Two other potentials of the class are the following extensions of $\hat{V}_3$ and $\hat{V}_4$ [1]:

$$V_3(x, y) = \hat{V}_3(x, y) + \eta y \qquad V_4(x, y) = \hat{V}_4(x, y) + \nu y^{-2}$$

where η, ν are constant parameters. The corresponding integrals are

$$I_3(x, y, p_x, p_y) = \hat{I}_3(x, y, p_x, p_y) - \eta b x^4$$
$$I_4(x, y, p_x, p_y) = \hat{I}_4(x, y, p_x, p_y) + 8a\nu x^4 y^{-2}.$$

It is the purpose of the present paper to show that the potential V_3 can be obtained from the potential V_4 by a procedure which involves the following steps: (i) a translation of one of the canonical variables, $y \to y + b/(4a)$; (ii) an appropriate choice of the parameters leading to the removal of the singularity at $a = 0$ introduced in the first step; and (iii) taking the limit $a \to 0$. The same procedure enables us to obtain results associated with potential V_3—for example, second integral of motion, separating variables, Lax representation, bi-Hamiltonian structure—from the corresponding ones for potential V_4.

We obtained this result when we attempted to generalize the recent finding by Ravoson *et al* [6], for potential V_3, and by Ravoson *et al* [7] and Romeiras [8], for potential V_4, of separating variables for systems with these potentials. We arrived at a potential V_N that includes both V_3 and V_4 as particular cases and gives the connection between the two. In fact the potential V_N is the result of the first two steps of the procedure described above.

In [6–8] the authors also obtained Lax representations [10] for system (1), (2) with potentials V_3 and V_4 by a method due to Fairbanks [11]. Using the same method we have obtained a Lax representation for system (1), (2) with potential V_N. The same result can be obtained by applying steps (i) and (ii) of our procedure to the Lax representations given in [7, 8] for system (1), (2) with potential V_4.

Ravoson [12] obtained a bi-Hamiltonian structure for system (1), (2) with potential V_3. Using his method we have obtained a bi-Hamiltonian structure for system (1), (2) with potential V_N, which for $b = 0$ gives a bi-Hamiltonian structure for system (1), (2) with potential V_4, a result that is new as far as we know.

It is known [1] that the potentials V_3 and V_4 have further integrable extensions in the class of systems we are considering: $V_3 + \mu x^{-2}$, $V_4 + \mu x^{-2}$, where μ is another constant parameter. Both these two potentials can be obtained from the corresponding integrable extension of V_N, $V_N + \mu x^{-2}$.

In [7] the authors obtained separating variables and a Lax representation for the integrable extension $V_4 + \mu x^{-2}$. Applying the first two steps of our procedure to these results, we have obtained separating variables and a Lax representation for the extension $V_N + \mu x^{-2}$. Letting $a \to 0$ we obtain a Lax representation for the integrable potential $V_3 + \mu x^{-2}$ which is new and constitutes an alternative to the 3×3 Lax representation for this system given by Blaszak and Rauch-Wojciechowski [9].

Ravoson [12] obtained what he called a (ρ, s)-bi-Hamiltonian structure for system (1), (2) with potential $V_3 + \mu x^{-2}$. We were able to obtain the corresponding structure for system (1), (2) with potential $V_4 + \mu x^{-2}$. Applying the first two steps of our procedure to this result we then obtained the (ρ, s)-bi-Hamiltonian structure for system (1), (2) with potential $V_N + \mu x^{-2}$.

The plan of the paper is as follows. In section 2 we describe the procedure to go from V_4 to V_3 through V_N; as an application we obtain I_3 from I_4 via I_N, the second integral of motion associated with the potential V_N. In section 3 we show how we arrived at the pair V_N, I_N. In section 4 we give a Lax representation and a bi-Hamiltonian structure for system (1), (2) with potential V_N. In section 5 we consider the integrable extension $V_N + \mu x^{-2}$, for which we give the separating variables, a Lax representation and a (ρ, s)-bi-Hamiltonian structure.

2. The procedure

The procedure connecting the potentials V_4 and V_3 can be described precisely in the following way. Define two auxiliary functions Φ and V_N by

$$\Phi(x, y; a, c, \nu) = V_4(x, y)$$

and

$$V_N(x, y; a, b, c, \beta_0, \beta_1) = \Phi\left(x, y + \frac{b}{4a}; a, \tilde{c}, -\frac{\tilde{\beta}_0}{128a^2}\right) - \frac{\tilde{\beta}_1}{16a}$$

where

$$\tilde{c} = c - \tfrac{3}{4}\gamma \qquad \tilde{\beta}_0 = \beta_0 - \beta_1\gamma - 8c\gamma^2 + 2\gamma^3 \qquad \tilde{\beta}_1 = \beta_1 + 16c\gamma - 6\gamma^2$$

and $\gamma = b^2/(2a)$ and β_0, β_1 are new constant parameters. Then

$$V_N(x, y; a, b, c, \beta_0, \beta_1) = a\left(x^4 + 6x^2y^2 + 8y^4\right) + by\left(3x^2 + 8y^2\right) + c\left(x^2 + 4y^2\right)$$
$$+\frac{y\left(-\beta_1 + 32bcy + 24b^2y^2\right)}{4(4ay + b)} + \frac{-\beta_0 + 2by\left(-\beta_1 + 16bcy + 8b^2y^2\right)}{8(4ay + b)^2} \qquad (4)$$

and

$$V_N(x, y; 0, b, d, 0, -2\eta b) = V_3(x, y).$$

By applying the same procedure we can obtain I_3 from I_4. Define two auxiliary functions Ψ and I_N by

$$\Psi(x, y, p_x, p_y; a, c, \nu) = I_4(x, y, p_x, p_y)$$

and

$$I_N(x, y, p_x, p_y; a, b, c, \beta_0, \beta_1) = \Psi\left(x, y + \frac{b}{4a}, p_x, p_y; a, \tilde{c}, -\frac{\tilde{\beta}_0}{128a^2}\right).$$

Then

$$I_N(x, y, p_x, p_y; a, b, c, \beta_0, \beta_1) = \left\{p_x^2 + 2x^2\left[a(x^2 + 2y^2) + by + c\right]\right\}^2$$
$$+4x^2(xp_y - 2yp_x)[a(xp_y - 2yp_x) - bp_x] - 2b^2x^4\left(x^2 + 2y^2\right)$$
$$-x^4\left\{\frac{8b^2y(by + c)}{4ay + b} + \frac{2a\beta_0 + b^2\left(-\beta_1 + 16bcy + 8b^2y^2\right)}{2(4ay + b)^2}\right\} \qquad (5)$$

and

$$I_N(x, y, p_x, p_y; 0, b, d, \beta_0, -2\eta b) = I_3(x, y, p_x, p_y).$$

The function V_N can be interpreted as another potential of the class we are considering that includes both V_3 and V_4 as particular cases. In fact V_N satisfies the identities

$$V_N(x, y; 0, b, d, \beta_0, \beta_1) = \hat{V}_3(x, y) - \frac{\beta_1}{2b}y - \frac{\beta_0}{8b^2}$$
$$V_N(x, y; a, 0, c, \beta_0, \beta_1) = \hat{V}_4(x, y) - \frac{\beta_0}{128a^2}y^{-2} - \frac{\beta_1}{16a}$$
$$V_N(x, y; a, b, c, \beta_0, \beta_1) = V_N\left(x, y + \frac{b}{4a}; a, 0, \tilde{c}, \tilde{\beta}_0, \tilde{\beta}_1\right)$$

that show that for $a = 0$ the potential V_N reduces to V_3 while for $a \neq 0$ it reduces to V_4 either directly (for $b = 0$) or after a translation of the variable y (for $b \neq 0$).

In this interpretation the function I_N is the second integral of motion associated with the Hamiltonian system (1), (2) with potential V_N. It satisfies the identities

$$I_N(x, y, p_x, p_y; 0, b, d, \beta_0, \beta_1) = \hat{I}_3(x, y, p_x, p_y) + \frac{\beta_1}{2}x^4$$
$$I_N(x, y, p_x, p_y; a, 0, c, \beta_0, \beta_1) = \hat{I}_4(x, y, p_x, p_y) - \frac{\beta_0}{16a}x^4y^{-2}$$
$$I_N(x, y, p_x, p_y; a, b, c, \beta_0, \beta_1) = I_N\left(x, y + \frac{b}{4a}, p_x, p_y; a, 0, \tilde{c}, \tilde{\beta}_0, \tilde{\beta}_1\right).$$

3. Derivation of the result

In this section we show how the pair V_N, I_N were obtained when we attempted to generalize the results of [6–8] on the existence of separating variables for system (1), (2) with potentials $\hat{V}_3$ and $\hat{V}_4$. These results are as follows.

Let (u, v) be two new variables defined by

$$u = -\sigma - \delta \qquad v = -\sigma + \delta \tag{6}$$

where

$$\sigma(x, y, p_x, p_y) = x^{-2}p_x^2 + S(x, y) \qquad \delta(x, y, p_x, p_y) = -x^{-2}\sqrt{I(x, y, p_x, p_y)} \tag{7}$$

and

$$\hat{V}_3: \quad S(x, y) = 2(by + d) \qquad I = \hat{I}_3$$
$$\hat{V}_4: \quad S(x, y) = 2\left[a\left(x^2 + 2y^2\right) + c\right] \qquad I = \hat{I}_4.$$

If we then eliminate the old variables (x, y, p_x, p_y) in terms of the new variables (u, v) and their time derivatives $(\dot{u}, \dot{v})$ from the equations

$$\tfrac{1}{2}\left(p_x^2 + p_y^2\right) + V(x, y) = H_* \qquad I(x, y, p_x, p_y) = I_*$$

where H_* and I_* are the constant values of the integrals of motion, we obtain the two differential equations

$$\dot{u}^2 = g_+(u) \qquad \dot{v}^2 = g_-(v) \tag{8}$$

where $g_\pm$ are cubic polynomial functions, defined by

$$\hat{V}_3: \quad g_\pm(z) = -2z^2(z + 4d) + 4b^2\left(2H_* \mp \sqrt{I_*}\right)$$
$$\hat{V}_4: \quad g_\pm(z) = -2z^2(z + 4c) + 8az\left(2H_* \mp \sqrt{I_*}\right).$$

We attempted to obtain a generalization of these results by proceeding in the following way: start with equations (8) but with $g_\pm$ of the more general form

$$g_\pm(z) = A(z)(2H \mp \sqrt{I}) + B(z) \tag{9}$$

where

$$A(z) = \sum_{k=0}^{n_A} a_k z^k \qquad B(z) = \sum_{k=0}^{n_B} b_k z^k$$

with the degrees of the polynomials n_A and n_B left unspecified; introduce the transformation of variables in the form given by (6) and (7), with S left unspecified; solve for H and I, with the requirement that H should be of the 'natural' form given by equation (2).

Adding and subtracting (8) and using (9) we obtain

$$[A(u)+A(v)](2H)-[A(u)-A(v)]\sqrt{I}+B(u)+B(v)-\dot{u}^2-\dot{v}^2=0$$
$$[A(u)-A(v)](2H)-[A(u)+A(v)]\sqrt{I}+B(u)-B(v)-\dot{u}^2+\dot{v}^2=0\,. \tag{10}$$

Noting that

$$\begin{aligned}A(u)+A(v)&=A(-\sigma-\delta)+A(-\sigma+\delta)\\&=2[a_0-a_1\sigma+a_2(\sigma^2+\delta^2)-a_3\sigma(\sigma^2+3\delta^2)+a_4(\sigma^4+6\sigma^2\delta^2+\delta^4)+\cdots]\\A(u)-A(v)&=A(-\sigma-\delta)-A(-\sigma+\delta)\\&=-2\delta[a_1-2a_2\sigma+a_3(3\sigma^2+\delta^2)-4a_4\sigma(\sigma^2+\delta^2)+\cdots]\end{aligned}$$

and similarly for B, and that

$$\delta=-x^{-2}\sqrt{I}\qquad\dot{\delta}=2x^{-3}\dot{x}\sqrt{I}$$

we conclude that in order to keep equations (10) linear in H and I we have to take $n_A=1$ and $n_B=3$. Equations (10) can then be written in the form

$$(2H)A(-\sigma)+\left(x^{-4}I\right)\left[\tfrac{1}{2}B''(-\sigma)-4\left(x^{-1}\dot{x}\right)^2-a_1x^2\right]=\dot{\sigma}^2-B(-\sigma)$$
$$(2H)a_1+\left(x^{-4}I\right)b_3=4\left(x^{-1}\dot{x}\right)\dot{\sigma}+x^2A(-\sigma)-B'(-\sigma)\,. \tag{11}$$

Noting that the 'natural' form of the Hamiltonian implies that

$$\dot{x}=p_x\qquad\dot{y}=p_y\qquad\dot{\sigma}=p_y\frac{\partial S}{\partial y}+\left(x^{-1}p_x\right)\left[-2\left(x^{-1}p_x\right)^2+x\frac{\partial S}{\partial x}-2x^{-1}\frac{\partial V}{\partial x}\right]$$

and solving equations (11) for H, we obtain

$$H=\frac{1}{2}p_x^2+\frac{1}{2D}\left(N_2p_y^2+N_1p_y+N_0\right) \tag{12}$$

where D, N_0, N_1 and N_2 are functions of (x, y, p_x) defined by

$$\begin{aligned}D&=a_0b_3+a_1\left[2r^2(b_3+2)+S_1-b_3S\right]\\N_0&=b_1b_2-b_0b_3+x^2(a_0a_1x^2-a_0b_2-a_1b_1)\\&\quad-S\left[a_1^2x^4-3(a_0b_3+a_1b_2)x^2+2b_1b_3+2b_2^2\right]-2b_3S^2(3a_1x^2-4b_2)-8b_3^2S^3\\&\quad+r^2(b_3+2)\left[S_2^2+2S(b_2-S_1)+2a_0x^2-2b_1\right]\\&\quad-2r^2\left[2r^2(b_3+2)+S_1-S_2\right]^2\\N_1&=2r\left(\frac{\partial S}{\partial y}\right)\left[4r^2(b_3+2)+2S_1+b_3S_2\right]\\N_2&=b_3\left(\frac{\partial S}{\partial y}\right)^2\end{aligned}$$

with r, S_1, S_2 given by

$$r=\frac{p_x}{x}\qquad S_1=3b_3S+a_1x^2-b_2\qquad S_2=x\frac{\partial S}{\partial x}-2x^{-1}\frac{\partial V}{\partial x}\,.$$

Comparison of equations (12) and (2) yields the compatibility equations

$$N_2=D\qquad N_1=0\qquad N_0=2DV\,.$$

The second of these equations forces

$$b_3=-2\qquad S_2=S_1$$

which, when substituted into the other two, and after some simplification, leads to a system of partial differential equations for S

$$\frac{\partial S}{\partial x} = \frac{a_1 x}{2} \qquad \left(\frac{\partial S}{\partial y}\right)^2 = -\frac{D}{2}$$

that can be integrated with the result

$$S(x, y) = \frac{a_1}{4}(x^2 + 2y^2) + 2by + \epsilon$$

where b and ϵ are two new constants that must satisfy

$$a_0 + \frac{a_1}{2}(b_2 + 4\epsilon) - 4b^2 = 0 .$$

Having obtained S one can easily complete the calculation of the integrals of motion H and I. If one introduces new constants a, c, β_0 and β_1, defined in terms of those introduced so far in this section by

$$a = \frac{a_1}{8} \qquad \beta_0 = b_0 + (2c - \epsilon)[b_1 - 4\epsilon(2c - \epsilon)]$$
$$c = \tfrac{1}{4}(b_2 + 6\epsilon) \qquad \beta_1 = b_1 - 2(2c - \epsilon)(2c + 3\epsilon)$$

we recover the pair V_N, I_N of equations (4) and (5).

The polynomial functions introduced in (9) can be written in the form

$$g_{\pm}(z) = 4[b^2 + 2a(z + \epsilon - 2c)](2H \mp \sqrt{I}) + G(z + \epsilon - 2c)$$

where G is defined by

$$G(z) = \beta_0 + \beta_1 z - 8cz^2 - 2z^3 . \tag{13}$$

Without loss of generality we can set $\epsilon = 2c$, as this is equivalent to a translation of the separating variables $z + \epsilon - 2c \to z$ that leaves the system (8) invariant. With this choice we finally obtain

$$S(x, y) = 2\left[a\left(x^2 + 2y^2\right) + by + c\right] \tag{14}$$

and

$$g_{\pm}(z) = 4(b^2 + 2az)(2H_* \mp \sqrt{I_*}) + G(z) \tag{15}$$

thus completing the calculation of the separating variables and the resulting differential equations (8) in these variables.

If one introduces the variables (p_u, p_v), defined by

$$p_u = \frac{\dot{u}}{8(b^2 + 2au)} \qquad p_v = \frac{\dot{v}}{8(b^2 + 2av)}$$

then the transformation from (x, y, p_x, p_y) to (u, v, p_u, p_v), as defined by (6) and (7) with $I = I_N$ and S given by (14) is canonical. In the canonical variables the Hamiltonian function $H_N = T + V_N$ and the second integral of motion I_N take the form

$$H_N = h(u, p_u) + h(v, p_v) \qquad -\tfrac{1}{2}\sqrt{I_N} = h(u, p_u) - h(v, p_v) \tag{16}$$

where

$$h(z. p_z) = 4(b^2 + 2az)p_z^2 - \frac{G(z)}{16(b^2 + 2az)} \tag{17}$$

with G given by (13).

4. Lax representation and bi-Hamiltonian structure

Following [6–8] and [11] we have obtained a Lax representation [10] with a spectral parameter λ

$$\dot{L}(\lambda) = [A(\lambda), L(\lambda)] \equiv A(\lambda)L(\lambda) - L(\lambda)A(\lambda)$$

for system (1), (2) with potential V_N by taking the Lax pair in the form of two 4×4 matrices

$$L(\lambda) = \begin{bmatrix} L_+(\lambda) & 0 \\ 0 & L_-(\lambda) \end{bmatrix} \qquad A(\lambda) = \begin{bmatrix} A_+(\lambda) & 0 \\ 0 & A_-(\lambda) \end{bmatrix} \tag{18}$$

where $L_\pm$ and $A_\pm$ are 2×2 matrices

$$L_\pm(\lambda) = \begin{bmatrix} V_\pm(\lambda) & U_\pm(\lambda) \\ W_\pm(\lambda) & -V_\pm(\lambda) \end{bmatrix} \qquad A_\pm(\lambda) = \begin{bmatrix} 0 & 1/2 \\ Y_\pm(\lambda) & 0 \end{bmatrix} \tag{19}$$

with elements

$$U_\pm(\lambda) = \lambda - z_\pm \qquad V_\pm(\lambda) = -\dot{U}_\pm(\lambda)$$
$$W_\pm(\lambda) = \frac{g_\pm(\lambda) - [V_\pm(\lambda)]^2}{U_\pm(\lambda)} \qquad Y_\pm(\lambda) = \frac{\dot{W}_\pm(\lambda)}{2V_\pm(\lambda)}.$$

Here,

$$z_+ = u \qquad z_- = v$$

are the separating variables introduced in (6) and $g_\pm$ are the two functions introduced in (15).

We have carried out the calculation of the matrix elements with the following result:

$$\begin{aligned}
U_\pm(\lambda) &= \lambda + r^2 + S \mp x^{-2}\sqrt{I_N} \\
V_\pm(\lambda) &= -2(4ay+b)p_y + 2r\left[r^2 + S + 4y(2ay+b)\right] \mp 2rx^{-2}\sqrt{I_N} \\
W_\pm(\lambda) &= -2\lambda^2 + 2\lambda\left(r^2 + S - 4c\right) + 8r(4ay+b)p_y \\
&\quad -4r^2\left[r^2 + S + 6y(2ay+b)\right] + 8y(2ay+b)S + 4b^2x^2 \\
&\quad \mp 2x^{-2}\left[\lambda - 2r^2 - 4y(2ay+b)\right]\sqrt{I_N} \\
Y_\pm(\lambda) &= -\lambda + 2\left(r^2 + S - 2c\right) \mp 2x^{-2}\sqrt{I_N}
\end{aligned}$$

where $r = p_x/x$ and S is defined by (14).

Following Ravoson's work [12] on system (1), (2) with potential $\hat{V}_3$ we have obtained a bi-Hamiltonian structure for system (1), (2) with potential V_N.

System (1), (2) can be written more succinctly in the form

$$\dot{\boldsymbol{x}} = J\nabla H$$

where $\boldsymbol{x} = (x_i)_{1\leqslant i\leqslant 4} = (x, y, p_x, p_y)$, ∇H denotes the gradient of H and $\boldsymbol{J}$ is the skew-symmetric 4×4 matrix with non-zero upper-diagonal elements $J_{13} = J_{24} = 1$; $\boldsymbol{J}$ is the structure matrix associated with the canonical Poisson bracket (see, for example, [13]). System (1), (2) is called bi-Hamiltonian if it can also be written in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{M}\nabla F$$

where F is a second Hamiltonian function and $\boldsymbol{M}$ is a skew-symmetric 4×4 matrix which satisfies the Jacobi identity, i.e. $\boldsymbol{M}$ is the structure matrix associated with another Poisson bracket.

We have found that system (1), (2) with potential V_N is bi-Hamiltonian with second Hamiltonian function $F = \sqrt{I_N}/2$ and structure matrix M with the following upper-diagonal elements:

$$M_{12} = \frac{1}{2F}(-xp_y) \qquad M_{13} = \frac{1}{2F}\{p_x^2 + 2x^2[a(x^2+6y^2)+3by+c]\}$$

$$M_{14} = \frac{1}{2F}x[2(x^2+4y^2)(4ay+b)+8y(2by+c)+\Delta(y)]$$

$$M_{23} = \frac{1}{2F}[p_xp_y - x^3(4ay+b)] \qquad M_{24} = -M_{13}$$

$$M_{34} = \frac{1}{2F}\{-4ax^3p_y + p_x[2(3x^2+4y^2)(4ay+b)+8y(2by+c)+\Delta(y)]\}$$

where

$$\Delta(y) = \frac{4by(3by+2c)}{4ay+b} + \frac{8b^2y(by+c)}{(4ay+b)^2} + \frac{2a\beta_0 + b^2(-\beta_1+16bcy+8b^2y^2)}{2(4ay+b)^3}.$$

5. Generalization

The potentials V_3 and V_4 have further integrable extensions in the class of systems we are considering in this paper [1]:

$$V_3^\mu(x,y) = V_3(x,y) + \mu x^{-2} \qquad V_4^\mu(x,y) = V_4(x,y) + \mu x^{-2}$$

where μ is another constant parameter. These two potentials can both be obtained from the corresponding integrable extension of V_N

$$V_N^\mu(x,y) = V_N(x,y) + \mu x^{-2}$$

with the associated second integral of motion

$$I_N^\mu(x,y,p_x,p_y) = I_N(x,y,p_x,p_y) + 4\mu x^{-2}\{p_x^2 + x^2S(x,y) + \mu x^{-2}\}$$

where S is given by (14).

It was shown by Ravoson *et al* [7] that the separability result for potential V_4 can be extended to potential V_4^μ. These authors also obtained a Lax representation for this system.

By applying the procedure described in section 2 to the results given in [7] we have obtained the following results valid for system (1), (2) with potential V_N^μ.

The separating variables are given by equations (6) and (7) with $I = I_N$ and S given by (14). The differential equations for (u, v) are now

$$\dot{u}^2 = \left[\frac{2R(u)}{R(u)+R(v)}\right]^2 g_+^\mu(u) \qquad \dot{v}^2 = \left[\frac{2R(v)}{R(u)+R(v)}\right]^2 g_-^\mu(v)$$

where

$$g_\pm^\mu(z) = 4(b^2+2az)\left[2H_*^\mu \mp R(z)\right] + G(z)$$

with

$$R(z) = \sqrt{I_*^\mu + 4\mu z}$$

and G is the function defined by (13). The momenta canonically conjugate to (u, v) are now given by

$$p_u = \frac{R(u)+R(v)}{2R(u)}\frac{\dot{u}}{8(b^2+2au)} \qquad p_v = \frac{R(u)+R(v)}{2R(v)}\frac{\dot{v}}{8(b^2+2av)}.$$

In the new canonical variables the Hamiltonian $H_N^\mu = T + V_N^\mu$ and the second integral of motion I_N^μ can be written in the form

$$H_N^\mu = h(u, p_u) + h(v, p_v) - \frac{\mu}{4}\,\frac{u-v}{h(u, p_u) - h(v, p_v)}$$

$$I_N^\mu = 4[h(u, p_u) - h(v, p_v)]^2 - 2\mu(u+v) + \frac{\mu^2}{4}\left[\frac{u-v}{h(u, p_u) - h(v, p_v)}\right]^2$$

with $h(z, p_z)$ defined by (17).

The Lax representation is of the form given by (18) and (19) with matrix elements

$$U_\pm(\lambda) = \lambda + r^2 + S \mp x^{-2}\sqrt{I_N^\mu + 4\mu\lambda} + 2\mu x^{-4}$$

$$V_\pm(\lambda) = -2(4ay+b)p_y + 2r\left[r^2 + S + 4y(2ay+b)\right] \mp 2rx^{-2}\sqrt{I_N^\mu + 4\mu\lambda} + 4\mu r x^{-4}$$

$$\begin{aligned} W_\pm(\lambda) = {} & -2\lambda^2 + 2\lambda\left(r^2 + S - 4c\right) + 8r(4ay+b)p_y \\ & -4r^2\left[r^2 + S + 6y(2ay+b)\right] + 8y(2ay+b)S + 4b^2x^2 \\ & \mp 2x^{-2}\left[\lambda - 2r^2 - 4y(2ay+b)\right]\sqrt{I_N^\mu + 4\mu\lambda} \\ & -4\mu x^{-4}\left[\lambda + 2r^2 - 4y(2ay+b)\right] \end{aligned}$$

$$Y_\pm(\lambda) = -\lambda + 2\left(r^2 + S - 2c\right) \mp 2x^{-2}\sqrt{I_N^\mu + 4\mu\lambda} - 8\mu x^{-4}\,.$$

Note that the separating variables (u, v) continue to satisfy the equations $U_+(u) = U_-(v) = 0$. By letting $a \to 0$ we obtain a Lax representation for system (1), (2) with potential V_3^μ which is new and an alternative to the 3×3 Lax representation given for this system by Blaszak and Rauch-Wojciechowski [9].

Ravoson [12] obtained what he called a (ρ, s)-bi-Hamiltonian structure for system (1), (2) with potential V_3^μ, which can be defined in the following way:

The quadruplet (M, F, J, H), where H is an integrable Hamiltonian defined in $\mathbb{R}^4$, F is the associated second integral of motion, J and M are 4×4 structure matrices associated with the canonical and another Poisson bracket, respectively, constitutes a (ρ, s)-bi-Hamiltonian structure if and only if there exist two smooth functions ρ and s defined in $\mathbb{R}^4$ such that the following equations are satisfied:

$$M\nabla F = \rho J \nabla H \qquad M\nabla H - J\nabla F = s\nabla H\,.$$

We have found that the quadruplet $\left(M, I_N^\mu, J, H_N^\mu\right)$ constitutes a (ρ, s)-bi-Hamiltonian structure when M has the upper-diagonal elements

$$\begin{aligned} & M_{12} = 0 \qquad M_{13} = -8ax^4 \qquad M_{23} = -4x^3(4ay+b) \\ & M_{14} = -4x^3(4ay+b) + k_a x^2 p_x \qquad M_{24} = -8f(x, y, p_x) + k_a x^2 p_y \\ & M_{34} = 4x^2[3(4ay+b)p_x - 4axp_y] - k_a x^2 \frac{\partial f(x, y, p_x)}{\partial x} \end{aligned}$$

where

$$k_a = 8\sqrt{-a} \qquad f(x, y, p_x) = \tfrac{1}{2}p_x^2 + x^2\left[a(x^2 + 6y^2) + 3by + c\right] + \mu x^{-2}$$

and the functions ρ, s are given by

$$\rho = M_{14}M_{23} - M_{13}M_{24} \qquad s = M_{13} + M_{24}\,.$$

In the limit $a \to 0$ we recover Ravoson's result for system (1), (2) with potential V_3^μ. The result for $b = 0$, that is, for system (1), (2) with potential V_4^μ, is new.

Acknowledgments

The author is grateful to one of the referees for calling his attention to V Ravoson's thesis [12]. This work was supported by the Junta Nacional de Investigação Científica e Tecnológica, under project no STRDA/P/CEN/528/92.

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